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AN ACCURATE NUMERICAL TECHNIQUE FOR SOLVING FRACTIONAL OPTIMAL CONTROL PROBLEMS

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In this article, we propose the shifted Legendre orthonormal polynomials for the numerical solution of the fractional optimal control problems that appear in several branches of physics and engineering. The Rayleigh-Ritz method for the necessary conditions of optimization and the operational matrix of fractional derivatives are used together with the help of the properties of the shifted Legendre orthonormal polynomials to reduce the fractional optimal control problem to solving a system of algebraic equations that greatly simplifies the problem. For confirming the efficiency and accuracy of the proposed technique, an illustrative numerical example is introduced with its approximate solution.

Key words: fractional optimal control problem, Legendre polynomials, operational matrix, Rayleigh-Ritz method, caputo derivatives.

1. INTRODUCTION

Due to the accuracy of *fractional calculus* in describing many mathematical, physical, and engineering phenomena in mechanical systems [1], chaos, solitons and fractals [2], finance [3], fluid-dynamics [4], solid mechanics [5], viscoelastic dampers [6] and others [7–10], it has become the focus of many researchers in recent years.

Orthogonal polynomials have received considerable attention in dealing with various problems. The main characteristics behind the approach using this technique is that it reduces such problems into that of solving a system of algebraic equations that greatly simplifies the problem. Some types of orthogonal polynomials have been used as basis functions of many techniques for solving fractional differential equations [11–14]. Recently, some types of orthogonal polynomials have been introduced as basis functions of the operational matrices of fractional derivatives and integrals used to solve ordinary and partial fractional differential equations [15–23].

The *optimal control problem* is a set of differential equations describing the paths of the control variables that minimize a function of the state and control variables. Optimal control problems can be found in many scientific and engineering applications, and it has become a very active and successful research area in recent years. The *fractional optimal control problem* is an optimal control problem in which the differential equations governing the dynamics of the system contain a fractional order derivative term. Recently, many researchers have been interested in studying the fractional optimal control problems and finding numerical solutions for them, see, for instance [24–31].

Our main aim in this paper is to develop an accurate numerical algorithm for solving the fractional optimal control problem. For that purpose, the shifted Legendre orthonormal polynomials are used as basis functions of the operational matrix of fractional derivatives. These orthonormal polynomials are used together with the Rayleigh-Ritz method to reduce the fractional optimal control problem into a problem consisting of solving a system of algebraic equations. That system can be solved by any iterative method.

This article is structured as follows. In Section 2, some properties of the shifted Legendre orthonormal polynomials are introduced, while in Section 3, we derive the operational matrix of fractional derivatives. In Section 4, the Rayleigh-Ritz method and the operational matrix of fractional derivatives have been used together with the help of the properties of the shifted Legendre orthonormal polynomials to solve the fractional optimal control problem. In Section 5, a numerical example and comparison between the results achieved using our numerical technique and those achieved using the numerical technique discussed in [30] is introduced. The conclusions are given in Section 6.

2. SHIFTED LEGENDRE ORTHONORMAL POLYNOMIALS

We assume that the Legendre polynomial of degree k is denoted by $P_k(z)$, and is defined on the interval $(-1, 1)$. $P_k(z)$ may be generated by the recurrence formulae:

$$\begin{aligned}
&P_{k+1}(z)=\frac{2k+1}{k+1}zP_k(z)-\frac{k}{k+1}P_{k-1}(z), \qquad 1\le k,\\
&P_0(z)=1, \qquad P_1(z)=z.
\end{aligned}$$

Introducing $z=2t-1$, the Legendre polynomials are defined on the interval $(0,1)$, which may be called shifted Legendre polynomials $P_k^*(t)$ and are generated using the following recurrence formulae

$$\begin{aligned}
&P_{k+1}^*(t)=\frac{2k+1}{k+1}(2t-1)P_k^*(t)-\frac{k}{k+1}P_{k-1}^*(t), \qquad 1\le k,\\
&P_0^*(t)=1, \qquad P_1^*(t)=2t-1.
\end{aligned}$$

The orthogonality relation is

$$\int_0^1 \mathrm{P}_j^*(t)\,\mathrm{d}t=\begin{cases}\dfrac{1}{2k+1} & \text{for} \quad j=k,\\ 0 & \text{for} \quad j\neq k.\end{cases} \tag{1}$$

The explicit analytical form of shifted Legendre polynomial $P_k^*(t)$ of degree k may be written as

$$\int_0^1 P_j^*(t)P_k^*(t)\mathrm{d}t=\begin{cases}\dfrac{1}{2k+1}, & \text{for} \quad j=k,\\ 0, & \text{for} \quad j\neq k,\end{cases} \tag{2}$$

Introducing the shifted Legendre orthonormal polynomials $P_k^{\mathring{a}}(t);\, P_k^{\mathring{a}}(t)\equiv\sqrt{2k+1}P_k^*(t),$ we have

$$\int_0^1 P_j^{\mathring{a}}(t)P_k^{\mathring{a}}(t)\mathrm{d}t=\begin{cases}1, & \text{for} \quad j=k,\\ 0, & \text{for} \quad j\neq k,\end{cases} \tag{3}$$

and

$$P_k^{\mathring{a}}(t)=\sqrt{2k+1}\sum_{i=0}^{k}(-1)^{k+i}\frac{(k+i)!}{(k-i)!\,(i!)^2}t^i. \tag{4}$$

Any square integrable function $y(t)$ defined on the interval $(0,1)$, may be expressed in terms of shifted Legendre orthonormal polynomials $P_k^{\mathring{a}}(t)$ as

$$y(t)=\sum_{k=0}^{\infty}y_kP_k^{\mathring{a}}(t),$$

from which the coefficients y_k are given by

$$y_k = \int_0^1 y(t) P_k^{\text{å}}(t)\mathrm{d}t, \;\; 0 \le k. \tag{5}$$

If we approximate $y(t)$ by the first $(N+1)$-terms, then we can write

$$y_N(t) = \sum_{k=0}^{N} y_k P_k^{\text{å}}(t), \tag{6}$$

which alternatively may be written in the matrix form:

$$y_N(t) = Y^T \Delta_N(t), \tag{7}$$

with

$$Y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \end{pmatrix}, \qquad \Delta_N(t) = \begin{pmatrix} P_0^{\text{å}}(t) \\ P_1^{\text{å}}(t) \\ \vdots \\ P_N^{\text{å}}(t) \end{pmatrix}. \tag{8}$$

3. OPERATIONAL MATRIX FOR FRACTIONAL DERIVATIVES

In this section, we will state and prove the fractional derivative of the shifted Legendre orthonormal polynomial.

THEOREM 3.1. *The fractional derivative of order* ν *of the shifted Legendre orthonormal polynomial vector* $\Delta_N(t)$ *is given by*

$$D^{\nu}\Delta_N(t) = D_{(\nu)}\Delta_N(t), \tag{9}$$

where

$$D^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_0^x (x-t)^{n-\nu-1} f^{(n)}(t)dt, \;\; n-1 < \nu \le n, \tag{10}$$

is the fractional derivative of the function $f(x)$ *in the Caputo sense, while* $D^{(\nu)}$ *is the* $(N+1)\times(N+1)$ *operational matrix of fractional derivative of order* ν *and is defined by*

$$D_{(\nu)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \Upsilon_\nu(n,0) & \Upsilon_\nu(n,1) & \Upsilon(n,2) & \cdots & \Upsilon_\nu(n,N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Upsilon_\nu(i,0) & \Upsilon_\nu(i,1) & \Upsilon_\nu(i,2) & \cdots & \Upsilon_\nu(i,N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Upsilon_\nu(N,0) & \Upsilon_\nu(N,1) & \Upsilon_\nu(N,2) & \cdots & \Upsilon_\nu(N,N) \end{pmatrix},$$

where

$$\Upsilon_{\nu}(i,j,k)=\sqrt{(2j+1)(2i+1)}\sum_{k=n}^{i}\sum_{l=0}^{j}\frac{(-1)^{i+j+k+l}(i+k)!\,(l+j)!}{(i-k)!\,k!\,\Gamma(k-\nu+1)(j-l)!\,(l!)^{2}(k+l-\nu+1)}. \tag{11}$$

Proof. Using (4) and (10), the fractional derivative of order ν for the shifted Legendre orthonormal polynomials $P_i^{å}(t)$ is given by

$$\begin{aligned} D^{\nu}P_i^{å}(t) &= \sqrt{2i+1}\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!\;(k!)^{2}}\;D^{\nu}t^{k} = \\ &= \sqrt{2i+1}\sum_{k=n}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!\;k!\;\Gamma(k-\nu+1)}\;t^{k-\nu}. \end{aligned} \tag{12}$$

Now we approximate $t^{k-\nu}$ by $N+1$ terms of shifted Legendre orthonormal polynomials $P_j^{å}(t)$ as:

$$t^{k-\nu}=\sum_{j=0}^{N}\mu_{kj}P_j^{å}(t), \tag{13}$$

where μ_{kj} is given as in Eq. (5) with $y(t)=t^{k-\nu}$, then

$$\begin{aligned} \mu_{kj} &= \int_0^1 t^{k-\nu}P_j^{å}(t)\mathrm{d}t = \\ &= \sqrt{2j+1}\sum_{l=0}^{j}(-1)^{j+l}\frac{(j+l)!}{(j-l)!\;(l!)^{2}}\int_0^1 t^{l+k-\nu}\mathrm{d}t = \\ &= \sqrt{2j+1}\sum_{l=0}^{j}(-1)^{j+l}\frac{(j+l)!}{(j-l)!\;(l!)^{2}\;(k-\nu+l+1)}. \end{aligned} \tag{14}$$

Employing Eqs. (12)-(14), we have

$$\begin{aligned} D^{\nu}P_i^{å}(t) &= \sqrt{2i+1}\sum_{k=n}^{i}\sum_{j=0}^{N}(-1)^{i+k}\frac{(i+k)!}{(i-k)!\;k!\;\Gamma(k-\nu+1)}\mu_{kj}P_j^{å}(t) = \\ &= \sum_{j=0}^{N}\Upsilon_{\nu}(i,j)P_j^{å}(t), \end{aligned} \tag{15}$$

where $\Upsilon(i,j)$ is given by Eq. (11). Finally, we can rewrite Eq. (15) in a vector form as

$$D^{\nu}P_i^{å}(t)=[\Upsilon_{\nu}(i,0),\,\Upsilon_{\nu}(i,1),\cdots,\,\Upsilon_{\nu}(i,j),\cdots,\,\Upsilon_{\nu}(i,N)]\Delta_N(t). \tag{16}$$

Equation (16) completes the proof.

3. THE NUMERICAL SCHEME

In this section, we use the properties of the shifted Legendre orthonormal polynomials together with the operational matrix of fractional integrals in order to solve the following fractional optimal control problem

$$\text{Min}\quad J=\int_{t_0}^{t_1}f(t,x(t),u(t))\mathrm{d}t, \tag{17}$$

subject to

$$D^{\nu} x(t) = g(t, x(t)) + b(t)u(t), \qquad n-1 < \nu \le n, b(t) \ne 0, \tag{18}$$

$$D^{(i)} x(t_0) = x_i, \qquad i = 0,1,\cdots,n-1. \tag{19}$$

First, we approximate $x(t)$ by the shifted Legendre orthonormal polynomials $P_i^{å}(t)$ as

$$x(t) = C^T \Delta_N(t), \tag{20}$$

where C is an unknown coefficients matrix that can be written as

$$C = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{pmatrix}. \tag{21}$$

Using Eq. (20), the dynamic constraint (18) may be written in the form

$$C^T D_{(\nu)} \Delta_N(t) = g(t, C^T \Delta_N(t)) + b(t)u(t). \tag{22}$$

The previous equation leads to

$$u(t) = \frac{1}{b(t)} (C^T D_{(\nu)} \Delta_N(t) - g(t, C^T \Delta_N(t))). \tag{23}$$

On the other hand, we can write the initial conditions (19) as in the form

$$C^T D_{(i)} \Delta_N(t_0) - x_i = 0, \qquad i = 0,1,\cdots,n-1. \tag{24}$$

or

$$t^i (C^T D_{(i)} \Delta_N(t_0) - x_i) = 0, \qquad i = 0,1,\cdots,n-1. \tag{25}$$

Now, we will merge the two previous equations as

$$u(t) = \frac{1}{b(t)} (C^T D_{(\nu)} \Delta_N(t) - g(t, C^T \Delta_N(t)) + \sum_{i=0}^{n-1} t^i (C^T D_{(i)} \Delta_N(t_0) - x_i)). \tag{26}$$

Using Eqs. (20) and (26), the performance index (17) may be written in the form

$$\begin{aligned} J_N[C^T] = & \int_{t_0}^{t_1} f(t, C^T \Delta_N(t), \\ & \frac{1}{b(t)} (C^T D_{(\nu)} \Delta_N(t) - g(t, C^T \Delta_N(t)) + \sum_{i=0}^{n-1} t^i (C^T D_{(i)} \Delta_N(t_0) - x_i)))\mathrm{d}t. \end{aligned} \tag{27}$$

Finally, after computing the previous integration, we can use the *Rayleigh-Ritz method* to show that the necessary conditions for the optimality of the performance index are

$$\frac{\partial J_N}{\partial c_0} = 0, \quad \frac{\partial J_N}{\partial c_1} = 0, \quad \cdots, \quad \frac{\partial J_N}{\partial c_N} = 0. \tag{28}$$

The system of algebraic equations introduced above can be solved by using any standard iteration method for the unknown coefficients c_j, $j = 0,1, \ldots, N$. Consequently, C given in (21) can be calculated.

4. NUMERICAL SIMULATION

In order to show the efficiency and accuracy of the proposed numerical technique, we applied it to solve an example that was introduced in [30] and we compared the results obtained in [30] with those achieved using our technique. Consider the following fractional optimal control problem

$$\text{Min } \; J = \int_0^1 \left[\left(x(t) - t^2 \right)^2 + \left(u(t) + t^4 - \frac{20\, t^{\frac{9}{10}}}{9\Gamma\left(9/10\right)} \right)^2 \right] \mathrm{d}t,$$

subjected to the dynamic constraints,

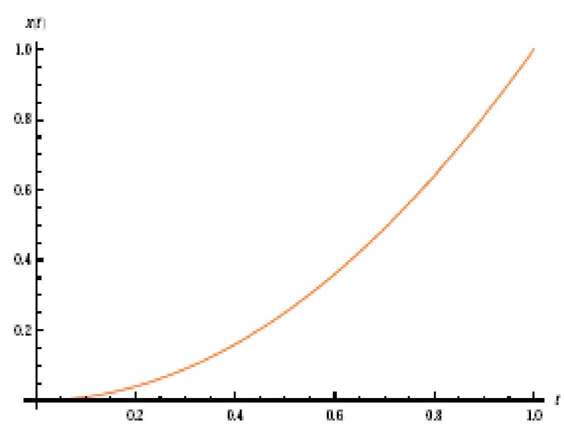
$$D^{1.1} x(t) = t^2 x(t) + u(t),$$
$$x(0) = \dot{x}(0) = 0.$$

For this problem $x(t) = t^2$ and $u(t) = \dfrac{20\, t^{\frac{9}{10}}}{9\Gamma\left(9/10\right)} - t^4$ minimize the performance index J and the minimum value is equal to 0. The minimum value of J achieved using our numerical technique is $5.81653 \times (10)^{-9}$ that is achieved at $N = 12$. In order to show that our numerical technique is better than that one introduced in [30], in Table 1 we compare the approximate values of the performance index J obtained using our approach with those obtained in [30] at different values of N. Also, in Fig. 1, we plot the exact and approximate values of the state and control variables, respectively, while Fig. 2 shows the error functions of the state and control variables at $N = 20$.

Table 1

Approximate values of *J* at different choices of *N*

N	Our method	Method in [30]
4	$4.76932 .10^{-6}$	$6.07530 .10^{-6}$
5	$1.47243 .10^{-6}$	$1.67255 .10^{-6}$
6	$5.37825 .10^{-7}$	$5.91532 .10^{-7}$
8	$1.06099 .10^{-7}$	$1.21966 .10^{-7}$
9	$5.44304 .10^{-8}$	$7.03371 .10^{-8}$



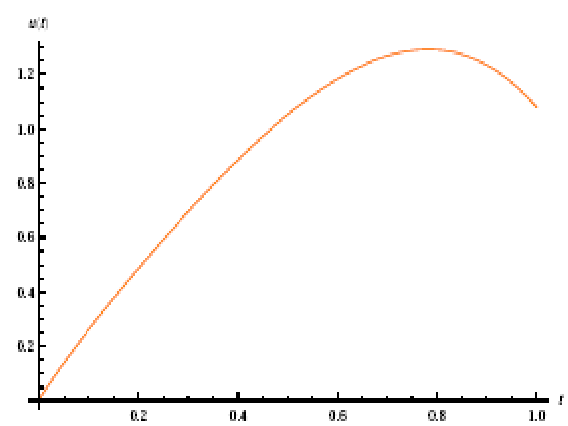


Fig. 1 – Exact and approximate state and control variables at *N* = 6.

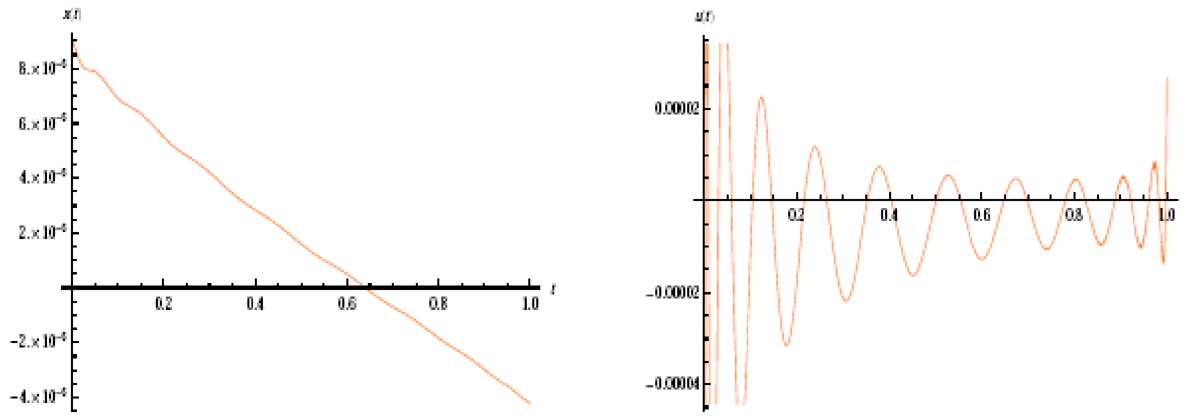


Fig. 2 – Error functions of the state and control variables at $N = 20$.

5. CONCLUSIONS

In this paper, we have proposed a new numerical technique based on the shifted Legendre orthonormal polynomials to approximate the solution of the fractional optimal control problem (17)-(19). The operational matrix of fractional derivatives and the properties of the shifted Legendre orthonormal polynomials are used together with the Rayleigh-Ritz method to reduce the fractional optimal control problem into a solution of a system of algebraic equations, greatly simplifying the problem. The fractional derivatives are described in the Caputo sense. The main advantage of the proposed algorithm is that adding a few terms of the shifted Legendre orthonormal polynomials, a good approximation of the exact solution of the problem was achieved. In order to clarify the validity and accuracy of our technique and to show that it is more accurate than that one introduced in [30], a numerical example is shown with its approximate solution and a comparison is made between our results and those obtained in Ref. [30].

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