Self-adjointness, Lie point symmetries, conservation laws and exact solutions for a two-dimensional generalized Grad-Shafranov equation

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Abstract

In this paper, Lie point symmetries and conservation laws for a two-dimensional generalized Grad-Shafranov equation (GGSE) governs an incompressible magnetohydrodynamic (MHD) flow are formulated. Lagrangians of the first and second order are derived. Several exact solutions to the latter equation are obtained. A construction for obtaining the solutions of the whole MHD system of incompressible flows is presented.

Keywords: Self-adjointness, Lie-point symmetries, Conservation laws, Grad-Shafranov equation.

1 Introduction

Study of Lie point symmetry is very important in the area of nonlinear partial differential equations (NPDEs), especially in integrable systems for the existence of infinitely many symmetries. To find the Lie point symmetry of a NPDE, a lot of powerful Lie group methods have been studied [1,2].

An important inclusion of symmetry in physics and mathematics is the existence of conservation laws. The concept of conservation law states that a certain measurable property of an isolated physical system does not change as the system evolves over time. It provides one of the basic principles in formulating and investigating models. The application of the conservation laws allows us to reach certain conclusions about the state of an object without the need for the details of all of the forces acting on it in a given situation throughout its motion. For example, sometimes, the existence of a large number of conservation laws of a given NPDE is a strong indication to its integrability [1,3].

An important theorem in mathematical physics known as Noether's theorem provides a systematic way of determining conservation laws for Lagrangians. For an Euler-Lagrange equation which arises out of a variational principle, this theorem constructed a one-to-one correspondence between a generalized variational symmetry of some functionals and a conservation law. Various generalizations of Noether's theorem were made by many researchers [1,3-9] to construct conservation laws. Ibragimov [1] proved that any Lie point, Lie-Bäcklund, and nonlocal symmetry of any system of differential equations provides a conservation law, provided that the number of equations is equal to the number of dependent variables. Ibragimov [1,3] introduced concepts of self-adjoint and quasi self-adjoint, where a general theorem on conservation laws was proved. The generalization of the concept of self-adjoint and quasi self-adjoint was introduced in [10]. It was found that a class of weak self-adjoint quasi-linear parabolic equations. The property of a differential equation to be weak self-adjoint is important for constructing conservation laws associated with symmetries of the differential equation. The general concept of nonlinear self-adjointness of differential equations was introduced by Ibragimov [8]. Many works have been carried out to obtain conditions for self-adjointness and conservation laws for nonlinear PDEs [11-17]. Self-adjointness, conservation laws and exact solutions for many types of PDEs were carried out by many authors [18-27]. A variational principle and conservation laws for coupled nonlinear schrodinger equations with variable coefficients and high nonlinearity were performed in [28].

In this paper, we formulate the Lagrangians and Lie point symmetries for a generalized Grad-Shafranov equation (GGSE) that governs the steady state ideal MHD with incompressible flows. The paper is organized as follows: In section 2, we present a formulation of the GGSE for the steady state ideal MHD flows. In section 3, we discuss the self-adjointness for the GGSE formulated in section 2. In sections 4 and 5, we formulate Lagrangians, Lie point symmetries and conservation laws for the same GGSE, and in sections 6 we obtain exact solutions for it. Finally, we summarize the results in section 7.

2 Formulation of a GGSE

The steady state ideal MHD flows are governed by the following set of equations, written in standard notations and convenient units [29-31]:

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P + \mathbf{J}\wedge\mathbf{B}, \tag{1}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{2}$$

$$\nabla\wedge\mathbf{E} = 0, \tag{3}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{4}$$

$$\mathbf{J} = \nabla\wedge\mathbf{B} \tag{5}$$

$$\mathbf{E} + \mathbf{v}\wedge\mathbf{B} = 0, \tag{6}$$

where $\rho, \mathbf{v}, \mathbf{E}, \mathbf{J}, \mathbf{B}$ and P stand as usual for the mass density, fluid velocity, electric field, electric current density, magnetic field and gas pressure, respectively.

For translational symmetric plasma with steady incompressible flow, the fields $\mathbf{B}, \mathbf{v}$ and $\mathbf{E}$ are expressed in terms scalar functions $\psi(x,y), \chi(x,y)$ and $\Phi(x,y)$ as:

$$\mathbf{B} = B_z\mathbf{e}_z + \mathbf{e}_z \wedge \nabla\psi,\ \mathbf{v} = v_z\mathbf{e}_z + \frac{1}{\rho}\mathbf{e}_z \wedge \nabla\chi,\ \mathbf{E} = -\nabla\Phi, \tag{7}$$

where the quantities B_z and v_z are the z-component for both of the magnetic field velocity field, respectively.

Ohm's law (6) is projected along the z-direction and yielding:

$$\chi = \chi(\psi), \Phi = \Phi(\psi) \tag{8}$$

The component of Eq. (6) perpendicular to a magnetic surface and the component of the momentum conservation Eq. (1) along z-direction yielding respectively:

$$(B_z\chi')/\rho - v_z = \Phi' \text{ and } B_z - \chi' v_z = \Lambda(\psi), \tag{9}$$

where Λ is a surface quantity. The dash denotes differentiation with respect to ψ.

From the incompressibility condition Eq. (2), the quantities B_z, v_z and ρ are surface quantities, i.e, $B_z = B_z(\psi)$, $v_z = v_z(\psi)$ and $\rho = \rho(\psi)$

Using the surface quantities $B_z(\psi)$, $v_z(\psi)$ and $\rho(\psi)$, the component of Eq. (1) along $\mathbf{B}$ yields the following expression for the pressure:

$$P = P_s(\psi) - \rho\chi'^2|\nabla\psi|^2/2, \tag{10}$$

where $P_s(\psi)$ is the static pressure.

With aid of Eqs. (7)-(10), the component of Eq. (1) perpendicular to a magnetic surface yields the following GGSE:

$$\left(1 - \frac{\chi'^2}{\rho}\right)\nabla^2\psi - \frac{1}{2}\left(\frac{\chi'^2}{\rho}\right)'|\nabla\psi|^2 + \left(P_s + \frac{B_z^2}{2}\right)' = 0. \tag{11}$$

Eq. (11) is a second order nonlinear NPDE for the poloidal magnetic flux function [3-5, 31].

From the analysis above we have seven surface quantities $\chi(\psi)$, $\Phi(\psi)$, $B_z(\psi)$, $v_z(\psi)$, $\rho(\psi)$, $P_s(\psi)$ and $\Lambda(\psi)$ five out of them are arbitrary.
Under the transformation:

$$\frac{du}{d\psi} = \sqrt{1 - \frac{\chi'^2}{\rho}}, \quad \chi'^2 < \rho, \tag{12}$$

Eq. (11) changes to

$$\nabla^2 u + \frac{d}{du}\left(P_s + \frac{B_z^2}{2}\right) = 0. \tag{13}$$

Therefore, Eq. (13) can be expressed mathematically as $\nabla^2 u = f(u)$.

3 Lagrangian and self-adjointness

For convenience we recall the following definitions [1,3,8,23]. Consider a nonlinear differential equation

$$F(x, u, u_{(1)}, u_{(2)}, ..., u_{(s)}) = 0, \tag{14}$$

with n independent variables $x = (x^1, ..., x^n)$ and dependent variable u, where $u_{(1)} = u_i$, $u_{(2)} = u_{ij}$, ... denote the sets of the partial derivatives of the first, second, etc. orders, $u_i = \frac{\partial u}{\partial x^i}$, $u_{ij} = \frac{\partial^2 u}{\partial x^i x^j}$.

Definition 1. The formal Lagrangian for Eq. (14) is defined by

$$L = \sum_{\beta=1}^{m} v^\beta F_\beta(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(s)}) \tag{15}$$

Definition 2. The adjoint equation to Eq. (14) is defined by

$$F^*(x, u, v, u_{(1)}, v_{(1)}, \cdots, u_{(s)}, v_{(s)}) = 0, \tag{16}$$

with

$$F^*(x, u, v, u_{(1)}, v_{(1)}, \cdots, u_{(s)}, v_{(s)}) = \frac{\delta(vF)}{\delta u}. \tag{17}$$

Definition 3. The Euler-Lagrange operator and equation are defined by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_i \frac{\partial}{\partial u_i} + D_i D_j \frac{\partial}{\partial u_{ij}} + \cdots, \tag{18}$$

$$\frac{\delta F}{\delta u}=\frac{\partial F}{\partial u}-D_i\frac{\partial F}{\partial u_i}+D_iD_j\frac{\partial F}{\partial u_{ij}}+\cdots=0, \tag{19}$$

where

$$D_i=\frac{\partial}{\partial x^i}+u_i\frac{\partial}{\partial u}+u_{ij}\frac{\partial}{\partial u_j}+u_{ijk}\frac{\partial}{\partial u_{jk}}+\cdots, \tag{20}$$

denotes the total differentiation.

Definition 4. A NPDE (14) is said to be strictly self-adjoint if its adjoint equation becomes equivalent to the original equation after the substitution $v=u$.

Definition 5. The concept of quasi-self-adjoint generalizes definition 4 by replacing the substitution in this definition by $v=\phi(u),\ \phi(u)\neq 0$

Definition 6. A NPDE (14) is said to be nonlinearly self-adjoint if its adjoint equation becomes equivalent to the original equation after the substitution $v=\phi$, where $\phi\neq 0$ is a nonzero function depending on the independent variables, the dependent variable as well as the partial derivatives of the dependent variable.

In agreement with [2] the formal Lagrangian for the GGSE equation

$$F\equiv u_{xx}+u_{yy}-f(u)=0, \tag{21}$$

is defined by

$$L=v\left[u_{xx}+u_{yy}-f(u)\right], \tag{22}$$

where v is new a dependent variable. Consequently, the adjoint equation to Eq. (21) has the form

$$F^*\equiv\frac{\delta L}{\delta u}=0. \tag{23}$$

where $\frac{\delta L}{\delta u}$ is the variational derivative of the second order Lagrangian in Eq. (22):

$$\frac{\delta L}{\delta u}=\frac{\partial L}{\partial u}+\frac{\partial^2}{\partial x^2}\left(\frac{\partial L}{\partial u_{xx}}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial L}{\partial u_{yy}}\right). \tag{24}$$

Taking into account the formal Lagrangian (22), the expanded form of the adjoint Eq. (23) to Eq. (21) reads

$$F^*=-vf'+v_{xx}+v_{yy}=0. \tag{25}$$

In what follows we discuss the adjointness of Eq. (21).

$$F^*|_{v=\phi(x,y,u)} = \mu\left[u_{xx} + u_{yy} - f(u)\right], \tag{26}$$

where μ is a regular undetermined coefficient.

$$\mu = \phi(x,y,u),\ \phi(x,y,u) \neq 0. \tag{27}$$

Using the differential consequences:

$$v_x = \phi_x + \phi_u u_x, \quad v_y = \phi_y + \phi_u u_y, \tag{28}$$

$$v_{xx} = \phi_{xx} + 2\phi_{xu}u_x + \phi_{uu}u_x^2 + \phi_u u_{xx}, \tag{29}$$

$$v_{yy} = \phi_{yy} + 2\phi_{yu}u_y + \phi_{uu}u_y^2 + \phi_u u_{yy}, \tag{30}$$

where subscripts denote partial derivatives, condition (26) is written as:

$$-\phi f'(u)+\phi_{xx}+2\phi_{xu}u_x+\phi_{uu}u_x^2+\phi_u u_{xx}+\phi_{yy}+2\phi_{yu}u_y+\phi_{uu}u_y^2+\phi_u u_{yy} = \mu[u_{xx}+u_{yy}-f(u)]. \tag{31}$$

The previous condition splits into the following system

$$\mu = \phi_u, \tag{32a}$$

$$2\phi_{xu} = 0, \tag{32b}$$

$$\phi_{uu} = 0, \tag{32c}$$

$$2\phi_{yu} = 0, \tag{32d}$$

$$-\phi f'(u) + \phi_{xx} + \phi_{yy} + \mu f(u) = 0. \tag{32e}$$

From Eqs. (32b,c,d), we get

$$\phi = c_1 u + h(x,y). \tag{33}$$

From Eq. (32e), we get

$$h_{xx} + h_{yy} + c_1 f(u) - (c_1 u + h)f'(u) = 0. \tag{34}$$

We try to find the function $h(x, y)$ taking into account Eq. (34). Consider the following cases:

Case 1: If $h = 0$,

the GGSE (Eq. (21)) is self-adjoint for $f(u) = au$ but it is quasi-self-adjoint for $c_1 \neq 1$ with $f(u) = au$.

Case 2: If $h = c_2 \neq 0$, Eq. (34) changes to

$$c_1 f(u) - (c_1 u + c_2) f'(u) = 0. \tag{35}$$

Integration of this equation yields

$$f(u) = b(u + \frac{c_2}{c_1}), \tag{36}$$

where b is an integration constant.
From Eq. (33) we deduce that the GGSE (Eq. (21)) is quasi-self-adjoint provided the function $f(u)$ be linear function of u.

Case 3: If $h(x, y)$ is any solution for Laplace's equation, Eq. (21) is nonlinear self-adjoint with $f(u) = 0$

Case 4: If $h(x, y)$ is any solution for Poisson's equation, Eq. (21) is nonlinear self-adjoint with $f(u) = const$.

4 Lagrangian for self-adjoint, quasi-self-adjoint and nonlinear self-adjoint

From the above analysis we find that when the function $f(u)$ is proportional to u; the GGSE Eq. (21) can be sef-adjoint or quasi-self-adjoint. And when it is a linear function of u; the GGSE Eq. (21) is nonlinear sef-adjoint. That is

$$f(u) = au + b \text{ with } \begin{cases} b = 0,\ a \text{ arbitrary} & \text{for self-adjoint and quasi-self-adjoint;} \\ a = 0,\ b \text{ arbitrary} & \text{for nonlinear self-adjoint.} \end{cases} \tag{37}$$

(A) For self-adjoint and quasi-self-adjoint GGSE

Using Eq. (37), Eq. (21) becomes

$$u_{xx} + u_{yy} - au = 0, \tag{38}$$

with the adjoint equation

$$v_{xx} + v_{yy} - av = 0. \tag{39}$$

The first order Lagrangian can be obtained as

$$L(u) = -\frac{1}{2}\left(u_x^2 + u_y^2 + au^2\right). \tag{40}$$

Proof:

Using Eq. (18), we have

$$\begin{aligned}
\frac{\delta L}{\delta u} &= \frac{\partial L}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial u_y}\right) \\
&= \left[\frac{\partial}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial u_y}\right)\right]\left[-\frac{1}{2}\left(u_x^2 + u_y^2 + au^2\right)\right] \\
&= -\frac{1}{2}\left(-2au\right) - \frac{1}{2}\frac{\partial}{\partial x}\left(-2u_x\right) - \frac{1}{2}\frac{\partial}{\partial y}\left(-2u_y\right) \\
&= u_{xx} + u_{yy} - au \equiv Eq.\,(38).
\end{aligned} \tag{41}$$

The second order Lagrangian is

$$L(u, v) = v(u_{xx} + u_{yy} - au). \tag{42}$$

Proof:

Using Eq. (18), we have

$$\begin{aligned}
\frac{\delta L}{\delta u} &= \left[\frac{\partial}{\partial u} + \frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial u_{xx}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial}{\partial u_{yy}}\right)\right]\left[v(u_{xx} + u_{yy} - au)\right] \\
&= -av + \frac{\partial^2}{\partial x^2}(v) + \frac{\partial^2}{\partial y^2}(v) \\
&= v_{xx} + v_{yy} - av \equiv Eq.\,(39),
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
\frac{\delta L}{\delta v} &= \frac{\partial L}{\partial v} + \frac{\partial^2}{\partial x^2}\left(\frac{\partial L}{\partial v_{xx}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial L}{\partial v_{yy}}\right) \\
&= \left[\frac{\partial}{\partial v} + \frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial v_{xx}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial}{\partial v_{yy}}\right)\right]\left[v(u_{xx} + u_{yy} - au)\right] \\
&= u_{xx} + u_{yy} - au \equiv Eq.\,(38).
\end{aligned} \tag{44}$$

(B) For nonlinear self-adjoint

Using Eq. (37), Eq. (21) becomes

$$u_{xx} + u_{yy} - b = 0, \tag{45}$$

with the adjoint equation

$$v_{xx} + v_{yy} = 0. \tag{46}$$

The first order Lagrangian is

$$L(u) = -\frac{1}{2}\left(u_x^2 + u_y^2\right) - bu. \tag{47}$$

Proof:

Using Eq. (18)

$$\begin{aligned}
\frac{\delta L}{\delta u} &= \left[\frac{\partial}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial u_y}\right)\right]\left[-\frac{1}{2}\left(u_x^2 + u_y^2\right) - bu\right] \\
&= -b - \frac{1}{2}\frac{\partial}{\partial x}\left(-2u_x\right) - \frac{1}{2}\frac{\partial}{\partial y}\left(-2u_y\right) \\
&= u_{xx} + u_{yy} - b \equiv Eq.\,(44).
\end{aligned} \tag{48}$$

The second order Lagrangian is

$$L(u, v) = v[u_{xx} + u_{yy} - b]. \tag{49}$$

Proof:

Using Eq. (18), we have

$$\begin{aligned}
\frac{\delta L}{\delta u} &= \frac{\partial L}{\partial u} + \frac{\partial^2}{\partial x^2}\left(\frac{\partial L}{\partial u_{xx}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial L}{\partial u_{yy}}\right) \\
&= \left[\frac{\partial}{\partial u} + \frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial u_{xx}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial}{\partial u_{yy}}\right)\right][v(u_{xx} + u_{yy} - b)] \\
&= v_{xx} + v_{yy} \equiv Eq.\,(45),
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
\frac{\delta L}{\delta v} &= \frac{\partial L}{\partial v} + \frac{\partial^2}{\partial x^2}\left(\frac{\partial L}{\partial v_{xx}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial L}{\partial v_{yy}}\right) \\
&= \left[\frac{\partial}{\partial v} + \frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial v_{xx}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial}{\partial v_{yy}}\right)\right][v(u_{xx} + u_{yy} - b)] \\
&= u_{xx} + u_{yy} - b \equiv Eq.\,(44).
\end{aligned} \tag{51}$$

5 Symmetries and conservation laws

A Lie-point symmetry of a PDE is an invertible transformation that leaves the equation unchanged [26,28]. The symmetry group of the GGSE will be generated by the vector field:

$$X = \xi^1(x, y, u)\frac{\partial}{\partial x} + \xi^2(x, y, u)\frac{\partial}{\partial y} + \eta(x, y, u)\frac{\partial}{\partial u}, \tag{52}$$

that is an infinitesimal generator with the prolongation operator:

$$X^* = X + \zeta_{11}\frac{\partial}{\partial u_{xx}} + \zeta_{22}\frac{\partial}{\partial u_{yy}}, \tag{53}$$

where

$$\zeta_1 = D_x(\eta) - u_x D_x(\xi^1) - u_y D_x(\xi^2), \tag{54a}$$

$$\zeta_2 = D_y(\eta) - u_x D_y(\xi^1) - u_y D_y(\xi^2), \tag{54b}$$

$$\zeta_{11} = D_x(\zeta_1) - u_{xx} D_x(\xi^1) - u_{xy} D_x(\xi^2), \tag{54c}$$

$$\zeta_{22} = D_y(\zeta_2) - u_{yx} D_y(\xi^1) - u_{yy} D_y(\xi^2). \tag{54d}$$

When we apply the operator (53) to Eq. (21), the invariant condition is

$$X^*(u_{xx} + u_{yy} - f(u)) = 0. \tag{55}$$

Using Eqs. (52)-(56) into Eq. (55) and equating the coefficients of partial derivatives to zero, we obtain a system of PDEs. After simple manipulations to that system we obtain a class of symmetries as: $\xi^1 = \alpha$, $\xi^2 = \beta$, $\eta = 0$ and f is arbitrary function of u. The combined system (14) and (17) admit the conservation law $D_i(C^i) = 0$ [24], where

$$\begin{aligned} C^i = \xi^i L + W \left[\frac{\partial L}{\partial u_i} - D_j\left(\frac{\partial L}{\partial u_{ij}}\right) + D_j D_k\left(\frac{\partial L}{\partial u_{ijk}}\right) - \cdots\right] \\ + D_j(W)\left[\frac{\partial L}{\partial u_{ij}} - D_k\left(\frac{\partial L}{\partial u_{ijk}}\right) + \cdots\right] + D_j D_k(W)\left[\frac{\partial L}{\partial u_{ijk}} - \cdots\right] + \cdots, \end{aligned} \tag{56}$$

with

$$W = \eta - \xi^i u_i. \tag{57}$$

For the considered case, we have

$$C^1 = \xi^1 L + W\left(\frac{\partial L}{\partial u_x} - D_x\frac{\partial L}{\partial u_{xx}} - D_y\frac{\partial L}{\partial u_{xy}}\right) + D_x W\left(\frac{\partial L}{\partial u_{xx}}\right) + D_y W\left(\frac{\partial L}{\partial u_{xy}}\right), \quad (58)$$

and

$$C^2 = \xi^2 L + W\left(\frac{\partial L}{\partial u_y} - D_x\frac{\partial L}{\partial u_{xy}} - D_y\frac{\partial L}{\partial u_{yy}}\right) + D_x W\left(\frac{\partial L}{\partial u_{xy}}\right) + D_y W\left(\frac{\partial L}{\partial u_{yy}}\right). \quad (59)$$

We apply Eqs. (58) and (59) to the above class of symmetries.

5.1 Conservation laws for self-adjoint and quasi-self-adjoint GGSE

1) First order Lagrangian:

Using Eqs. (58) and (59), we get

$$\begin{aligned} C^1 &= -\frac{\alpha}{2}\left(u_x^2 + u_y^2 + au^2\right) - (\alpha u_x + \beta u_y)(-u_x) - (\alpha u_{xx} + \beta u_{xy})(0) \\ &= \frac{\alpha}{2}\left(u_x^2 - u_y^2 - au^2\right) + \beta u_x u_y \end{aligned} \quad (60a)$$

$$\begin{aligned} C^2 &= -\frac{\beta}{2}\left(u_x^2 + u_y^2 + au^2\right) - (\alpha u_x + \beta u_y)(-u_y) - (\alpha u_{xy} + \beta u_{yy})(0) \\ &= \frac{\beta}{2}\left(u_y^2 - u_x^2 - au^2\right) + \alpha u_x u_y. \end{aligned} \quad (60b)$$

Proof:

$$\begin{aligned} D_i(C^i) &= D_x(C^1) + D_y(C^2) \\ &= -\frac{\alpha}{2}\left(-2u_x u_{xx} + 2u_y u_{xy} + 2auu_x\right) + \beta\left(u_{xx}u_y + u_x u_{xy}\right) \\ &\quad - \frac{\beta}{2}\left(-2u_x u_{xy} - 2u_y u_{yy} + 2auu_y\right) + \alpha\left(u_x u_{yy} + u_{xy}u_y\right) \\ &= (\alpha u_x + \beta u_y)(u_{xx} + u_{yy} - au) \\ &= 0. \end{aligned} \quad (61)$$

2) Second order Lagrangian:

Using Eqs. (58) and (59), we get

$$\begin{aligned} C^1 &= \alpha v (u_{xx} + u_{yy} - au) - (\alpha u_x + \beta u_y)(-v_x) - (\alpha u_{xx} + \beta u_{xy}) v \\ &= \alpha v (u_{yy} - au) + v_x (\alpha u_x + \beta u_y) - \beta v u_{xy} \end{aligned} \quad (62a)$$

$$\begin{aligned} C^2 &= \beta v (u_{xx} + u_{yy} - au) - (\alpha u_x + \beta u_y)(-v_y) - (\alpha u_{xy} + \beta u_{yy}) v \\ &= \beta v (u_{xx} - au) + v_y (\alpha u_x + \beta u_y) - \alpha v u_{xy}. \end{aligned} \quad (62b)$$

Proof:

$$
\begin{aligned}
D_i(C^i) &= D_x(C^1) + D_y(C^2) \\
&= -\frac{\alpha}{2}\left(-2u_x u_{xx} + 2u_y u_{xy} + 2auu_x\right) + \beta\left[u_{xx}u_y + u_x u_{xy}\right) \\
&\quad -\frac{\beta}{2}\left(-2u_x u_{xy} - 2u_y u_{yy} + 2auu_y\right) + \alpha\left(u_x u_{yy} + u_{xy}u_y\right) \\
&= \left(\alpha u_x + \beta u_y\right)\left(u_{xx} + u_{yy} - au\right) \\
&= 0.
\end{aligned} \tag{63}
$$

5.2 Conservation laws for nonlinear self-adjoint GGSE

1) First order Lagrangian:

$$
\begin{aligned}
C^1 &= -\frac{\alpha}{2}\left(u_x^2 + u_y^2\right) - \alpha bu - \left(\alpha u_x + \beta u_y\right)\left(-u_x\right) \\
&= \frac{\alpha}{2}\left(u_x^2 - u_y^2\right) - \alpha bu + \beta u_x u_y
\end{aligned} \tag{64a}
$$

$$
\begin{aligned}
C^2 &= -\frac{\beta}{2}\left(u_x^2 + u_y^2\right) - \beta bu - \left(\alpha u_x + \beta u_y\right)\left(-u_y\right) \\
&= \frac{\beta}{2}\left(u_y^2 - u_x^2\right) - \beta bu + \alpha u_x u_y.
\end{aligned} \tag{64b}
$$

Proof:

$$
\begin{aligned}
D_i(C^i) &= D_x(C^1) + D_y(C^2) \\
&= \frac{\alpha}{2}\left(2u_x u_{xx} - 2u_y u_{xy}\right) - \alpha bu_x + \beta\left(u_x u_{xy} + u_{xx}u_y\right) \\
&\quad + \frac{\beta}{2}\left(2u_y u_{yy} - 2u_x u_{xy}\right) - \beta bu_y + \alpha\left(u_x u_{yy} + u_{xy}u_y\right) \\
&= \left(\alpha u_x + \beta u_y\right)\left(u_{xx} + u_{yy} - b\right) \\
&= 0.
\end{aligned} \tag{65}
$$

2) Second order lagrangian:

$$
\begin{aligned}
C^1 &= \alpha v\left(u_{xx} + u_{yy} - b\right) - \left(\alpha u_x + \beta u_y\right)\left(-v_x\right) - \left(\alpha u_{xx} + \beta u_{xy}\right)v \\
&= \alpha v\left(u_{yy} - b\right) + v_x\left(\alpha u_x + \beta u_y\right) - \beta v u_{xy}
\end{aligned} \tag{66a}
$$

$$
\begin{aligned}
C^2 &= \beta v\left(u_{xx} + u_{yy} - b\right) - \left(\alpha u_x + \beta u_y\right)\left(-v_y\right) - \left(\alpha u_{xy} + \beta u_{yy}\right)v \\
&= \beta v\left(u_{xx} - b\right) + v_y\left(\alpha u_x + \beta u_y\right) - \alpha v u_{xy}.
\end{aligned} \tag{66b}
$$

$$
\begin{aligned}
D_i(C^i) &= D_x(C^1) + D_y(C^2) \\
&= \alpha\left[vu_{xyy} + v_x(u_{yy} - b)\right] + v_{xx}\left(\alpha u_x + \beta u_y\right) + v_x\left(\alpha u_{xx} + \beta u_{xy}\right) \\
&\quad - \beta\left(vu_{xxy} + v_x u_{xy}\right) + \beta\left[vu_{xxy} + v_y(u_{xx} - b)\right] + v_{yy}\left(\alpha u_x + \beta u_y\right) \\
&\quad + v_y\left(\alpha u_{xy} + \beta u_{yy}\right) - \alpha\left(vu_{xyy} + v_y u_{xy}\right) \\
&= \left(\alpha v_x + \beta v_y\right)\left(u_{xx} + u_{yy} - b\right) + \left(\alpha u_x + \beta u_y\right)\left(v_{xx} + v_{yy}\right) \\
&= 0.
\end{aligned}
\tag{67}
$$

6 Exact solutions

Using the class of symmetries introduced in section 5, the characteristic equation for Eq. (55) can be determined as:

$$\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{du}{0}. \tag{68}$$

The general solution of this equation is

$$u = \phi(\beta x + \alpha y) \equiv \phi(\tau), \quad \tau = \beta x + \alpha y, \tag{69}$$

where ϕ is arbitrary function of τ with

$$f(\phi) = (\alpha^2 + \beta^2)\frac{d^2\phi}{d\tau^2}. \tag{70}$$

Table (1) shows some solutions to Eq. (55) for some choices of the function $f(u)$. Using

Table 1: Some solutions for Eq. (55) where $A, k, n, A_i, i = 1 \cdots 8$ are real constants.

u	$\frac{d^2\phi}{d\tau^2}$	$f(u)$
$(\beta x + \alpha y)^k + A$	$k(k-1)\tau^{k-2}$	$(\alpha^2+\beta^2)[k(k-1)(u-A)^{\frac{k-2}{k}}]$
$\sin^n(\beta x + \alpha y) + A_1$	$n(n-1)\sin^{n-2}\tau - n^2\sin^n\tau$	$(\alpha^2+\beta^2)[n(n-1)(u-A_1)^{\frac{n-2}{n}} - n^2(u-A_1)]$
$\cos^n(\beta x + \alpha y) + A_2$	$n(n-1)\cos^{n-2}\tau - n^2\cos^n\tau$	$(\alpha^2+\beta^2)[n(n-1)(u-A_2)^{\frac{n-2}{n}} - n^2(u-A_2)]$
$\sinh^n(\beta x + \alpha y) + A_3$	$n(n-1)\sinh^{n-2}\tau + n^2\sinh^n\tau$	$(\alpha^2+\beta^2)[n(n-1)(u-A_3)^{\frac{n-2}{n}} + n^2(u-A_3)]$
$\cosh^n(\beta x + \alpha y) + A_4$	$n^2\cosh^n\tau - n(n-1)\cosh^{n-2}\tau$	$(\alpha^2+\beta^2)\left[n^2(u-A_4) - n(n-1)(u-A_4)^{\frac{n-2}{n}}\right]$
$sech^n(\beta x + \alpha y) + A_5$	$n^2 sech^n\tau - n(n+1)sech^{n+2}\tau$	$(\alpha^2+\beta^2)\left[n^2(u-A_5) - n(n+1)(u-A_5)^{\frac{n-2}{n}}\right]$
$\tanh^n(\beta x + \alpha y) + A_6$	$n(n-1)\tanh^{n-2}\tau - 2n\tanh^n\tau$	$(\alpha^2+\beta^2)\Big[n(n-1)(u-A_6)^{\frac{n-2}{n}} - 2n(u-A_6)$
	$+(n+1)\tanh^{n+2}\tau$	$+(n+1)(u-A_6)^{\frac{n+2}{n}}\Big]$
$e^{(\beta x + \alpha y)} + A_7$	e^τ	$(\alpha^2+\beta^2)(u-A_7)$
$\ln(\beta x + \alpha y) + A_8$	$-\tau^{-2}$	$-(\alpha^2+\beta^2)e^{-2(u-A_8)}$

$\bar{B} = \mathbf{e}_z \wedge \nabla u$, $Q = 1/\sqrt{1 - \frac{\chi'^2}{\rho}}$ and Eqs. (5), (7) and (10), we obtain:

$$\mathbf{B} = B_z \mathbf{e}_z + Q\bar{B}, \tag{71}$$

$$\mathbf{J} = j_z \mathbf{e}_z - QB'_z \bar{B}, \tag{72}$$

$$\mathbf{v} = v_z \mathbf{e}_z \pm \frac{\sqrt{Q^2 - 1}}{\sqrt{\rho}} \bar{B}, \tag{73}$$

$$P = P_s - (Q^2 - 1)|\nabla u|^2, \tag{74}$$

$$\mathbf{E} = -\Phi' Q \nabla u. \tag{75}$$

For example we apply Eqs. (71)-(75) to the solution $u = \sin^k(\beta x + \alpha y) + A_1$ shown in the second row of Table (1), hence we obtain

$$\mathbf{B} = \left\{ -\frac{\sqrt[3]{2}\alpha k \cos(\beta x + \alpha y) \sin^{k-1}(\beta x + \alpha y)}{\sqrt[3]{2 - 3\left(A_1 + \sin^k(\beta x + \alpha y)\right)}}, \frac{\sqrt[3]{2}\beta k \cos(\beta x + \alpha y) \sin^{k-1}(\beta x + \alpha y)}{\sqrt[3]{2 - 3\left(A_1 + \sin^k(\beta x + \alpha y)\right)}}, B_z \right\}, \tag{76}$$

$$\mathbf{J} = j_z \mathbf{e}_z + \left\{ \frac{\sqrt[3]{2}\alpha k \cos(\beta x + \alpha y) \sin^{k-1}(\beta x + \alpha y)}{\sqrt[3]{2 - 3\left(A_1 + \sin^k(\beta x + \alpha y)\right)}}, -\frac{\sqrt[3]{2}\beta k \cos(\beta x + \alpha y) \sin^{k-1}(\beta x + \alpha y)}{\sqrt[3]{2 - 3\left(A_1 + \sin^k(\beta x + \alpha y)\right)}}, B'_z \right\}, \tag{77}$$

$$\mathbf{v} = k \cos(\beta x + \alpha y) \sin^{k-1}(\beta x + \alpha y) \left(\pm \frac{\sqrt{\frac{2^{2/3}}{\left(2 - 3\left(A_1 + \sin^k(\beta x + \alpha y)\right)\right)^{2/3}} - 1}}{\sqrt{\rho}} \right) \{-\alpha, \beta, 0\}, \tag{78}$$

$$P = P_s + k^2 \left(\alpha^2 + \beta^2\right) \cos^2(\beta x + \alpha y) \sin^{2(k-1)}(\beta x + \alpha y) \left(1 - \frac{2^{2/3}}{\left(2 - 3\left(A_1 + \sin^k(\beta x + \alpha y)\right)\right)^{2/3}} \right), \tag{79}$$

$$\mathbf{E} = \left\{ -\frac{\sqrt[3]{2}\beta k \cos(\beta x + \alpha y) \sin^{k-1}(\beta x + \alpha y)}{\sqrt[3]{2 - 3\left(A_1 + \sin^k(\beta x + \alpha y)\right)}}, -\frac{\sqrt[3]{2}\alpha k \cos(\beta x + \alpha y) \sin^{k-1}(\beta x + \alpha y)}{\sqrt[3]{2 - 3\left(A_1 + \sin^k(\beta x + \alpha y)\right)}}, 0 \right\}. \tag{80}$$

Eqs. (76)-(80) show exact solutions to the full MHD system (1)-(6).

Figures (1) and (2) with values of parameters listed in their captions show soliton-like configurations for the magnetic flux function described by the solution $u = \sin^k(\beta x + \alpha y) + A_1$ shown at the second row of Table (1).

7 Summary

In this paper, we have investigated the symmetric equilibrium of ideal MHD incompressible flows in a Cartesian geometry. The MHD equilibrium is governed by an elliptic second-order NPDE for the poloidal magnetic flux function. Lie point symmetries and conservation laws for an incompressible MHD flow governed by a two-dimensional GGSE are formulated. Several exact solutions to the latter equation are obtained. The self-adjointness of the GGSE is discussed where we noted that it can be self-adjoint or quasi-self-adjoint or nonlinear self-adjoint according to the form of the function $f(u)$ appears in (see Eq. (21)). First and second order Lagrangians for the GGSE are proved. A construction for obtaining the solutions of the whole MHD system of incompressible flows is explained via the obtained solutions of the GGSE.

The obtained solutions cover previously configurations and include new considerations on the nonlinearity of magnetic flux stream variables. In [29] the velocity was taken to be parallel to the magnetic flux function and the Alfvénic mach number was taken to be a function of a dimensionless horizontal distance $M^2 = M^2(x)$, beside the velocity and the magnetic fields have an exponential dependence on Z. A solution was presented in [32] in terms of $\tanh \xi$, $\tan \xi$ and $\cot \xi$ where $\xi = x + \alpha y + \beta t$ can be obtained as a special case comparing with our results. In [33] the gas pressure was taken to be an isotropic while in the present paper we did not provide this assumption on the pressure. In [34] the travelling wave method was used to get solutions of incompressible ideal Hall MHD, also the velocity and magnetic field were taken parallel to the wave vector. Numerical solutions were presented in [35, 36]. The present paper is devoted to the investigation of generalized forms of the GGSE that describes symmetric plasma equilibria in the presence of poloidal and axial incompressible flows via Lie-point symmetries and conservation laws.

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Figure Captions

FIG. 1. Three-dimensional plots of the magnetic surfaces describe plasma in symmetric systems. The magnetic surfaces are plotted for the solution shown at the second row of Table (1). The values of parameters used are: $A_1 = 5, n = 5, \alpha = 8, \beta = 9$.

FIG. 2. Three-dimensional magnetic surfaces for the solution shown at the second row of Table (1). The same values of parameters in Fig. (1) are used with $n = 6$.