Indian J Phys (June 2013) 87(6):555–565
DOI 10.1007/s12648-013-0255-y

ORIGINAL PAPER

(G'/G)-expansion method for two-dimensional force-free magnetic fields described by some nonlinear equations

M A Abdelkawy[1]* and A H Bhrawy[1,2]

[1]Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef 62511, Egypt

[2]Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia

Received: 03 November 2012 / Accepted: 18 January 2013 / Published online: 12 February 2013

Abstract: In this paper three nonlinear force-free magnetic field equations such as the Liouville equation, Sine-Poisson equation and Sinh-Poisson equation are studied by (G'/G)-expansion method and exact periodic solutions are extracted. In all these cases the ratio of the current density and the magnetic field is not constant e.g. as happens in the solar atmosphere.

Keywords: (G′/G)-expansion method; Force-free magnetic field; Magnetostatic equation; Plasmas; Liouville equation; Sine-Poisson equation; Sinh-Poisson equation

PACS Nos.: 02.30.Jr; 07.05.Bx; 87.50.Mn; 04.20.Jb; 41.20.Gz

1. Introduction

The study of nonlinear evolution equations (NLEEs) appear everywhere in applied mathematics and theoretical physics including engineering sciences and biological sciences. These equations play a key role in describing key scientific phenomena. A force-free magnetic field problem is one of the very important fields, which arises as a special case from the magnetostatic equation in plasmas; which can be written in the form:

$$\nabla^2\phi + \left(\frac{B_z^2}{2}\right)' = 0, \tag{1}$$

where ϕ is denoted as flux function and B_z is the component of the magnetic field along the direction z, which is identified by the unit vector e_z. Moreover, the symbol $'$ denotes derivative with respect to the argument ϕ and B_z is an arbitrary function of ϕ [1]. A general form of the energy integral of semicircularly arched force-free magnetic fields has been studied based on the energy principle in [2–4]. Amari et al. [5] have studied different mathematical problems of the solar coronal magnetic field as a force-free magnetic field. Boulmezaoud and Amari [6] have used finite-element method for computing some nonlinear problems of force-free fields. Romashets and Vandas [7] have studied force-free magnetic field in a cylindrical flux rope without a constant alpha . Khater et al. [1] have studied several problems of force-free magnetic field by using generalized tanh method. In all those cases, the ratio of the current density and the magnetic field is not constant.

Recently, a number of methods have been proposed, namely mapping method [8], Jacobi elliptic function method [9], the homogeneous balance method [10–12], the hyperbolic tangent function expansion method [13–15], the trial function method [16, 17], the nonlinear transformation method [18, 19] and sine-cosine method [20]. However, these methods can only obtain the shock and solitary wave solutions and can not obtain the periodic solutions of nonlinear wave equations. Although Porubov and others [21–23] have obtained some exact periodic solutions to some nonlinear wave equations, using the Weierstrass elliptic function.

Wang et al. [24] have introduced the (G'/G)-expansion method for a reliable treatment of the nonlinear wave equations. The (G'/G)-expansion method plays an important role to find the exact solutions of nonlinear wave equations in the nonlinear problems [25–30].

In this article, we find the exact solution of two-dimensional force-free magnetic fields described by Liouville equation, Sinh-Poission equation and Sine-Poission equation using the (G'/G)-expansion method. It is shown here that the main merits of the (G'/G)-expansion method

*Corresponding author, E-mail: melkawy@yahoo.com

over the other methods are that it gives more general solutions with some free parameters which, by suitable choice of the parameters, turn out to be some known solutions gained by the existing methods.

2. Description of the (G'/G)-expansion method

In this section we describe the (G'/G)-expansion method for finding travelling wave solutions of nonlinear evolution equations. Suppose that a nonlinear equation, say in two independent variables x and y, is given by

$$P(u, u_y, u_x, u_{yy}, u_{xy}, u_{xx}, \ldots) = 0, \tag{2}$$

where $u = u(x, y)$ is an unknown function, P is a polynomial in $u = u(x, y)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the (G'/G)-expansion method.

i. We seek traveling wave solutions of Eq. (2) by taking

$$u(x,t) = U(\zeta), \quad \zeta = kx + vy,$$

and transform Eq. (2) to the ordinary differential equation

$$Q(U, U', U'', U''', \ldots) = 0, \tag{3}$$

where prime denotes the derivative with respect to ζ.

ii. Then we integrate Eq. (3) term by term one or more times. This yields constant(s) of integration. For simplicity, the integration constant(s) can be set to zero.

iii. We introduce the solution U(ζ) of Eq. (3) in the finite series form

$$U(\zeta) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \tag{4}$$

where a_i are real constants with $a_m \neq 0$ to be determined, m is a positive integer to be determined. The function $G(\zeta)$ is the solution of the auxiliary linear ordinary differential equation

$$G'' + \beta G' + \mu G = 0, \tag{5}$$

iv. We determine m, this usually can be accomplished by balancing the linear term(s) of highest order with the highest order nonlinear term(s) in Eq. (3).

v. Then substitution of Eqs. (4) and (5) into Eq. (3) yields an algebraic equation involving powers of (G'/G). Equating the coefficients of each power of (G'/G) to zero gives a system of algebraic equations for a_i, β, μ, k and ν. Then, we solve the system with the aid of a computer algebra system , such as Mathematica, to determine these constants. On the other hand, depending on the sign of the discriminant $\Delta = \beta^2 - 4\mu$, the solutions of Eq. (5) are well known to us. So, as a final step, we can obtain exact solutions of the given Eq. (2).

3. Two-dimensional force-free magnetic fields described by Liouville equation

Considering the choice $B_z = e^{\phi}$, Eq. (1) turns into the nonlinear Liouville equation

$$\nabla^2 \phi = -e^{2\phi}, \tag{6}$$

and taking the transformation

$$e^{2\phi} = u,$$

Equation (6) becomes

$$(u_x)^2 + (u_y)^2 - uu_{xx} - uu_{yy} - 2u^3 = 0. \tag{7}$$

Using the wave variable

$$\zeta = kx + vy, \quad u(x, y) = U(\zeta), \tag{8}$$

carries the partial differential equation (PDE) given by Eq. (7) into the ordinary differential equation (ODE)

$$(k^2 + v^2)UU'' - (k^2 + v^2)(U')^2 + 2U^3 = 0. \tag{9}$$

Balancing the term UU'' with the term U^3 we obtain $N = 2$ then

$$U(\zeta) = \sum_{i=0}^{2} a_i (G'/G)^i = a_0 + a_1(G'/G) + a_2(G'/G)^2. \tag{10}$$

Substituting Eq. (10) into Eq. (9) and comparing the coefficients of each power of (G'/G) in both sides, to get an over-determined system of nonlinear algebraic equations with respect to a_i, β, μ, k and ν. Solving the over-determined system of nonlinear algebraic equations by using Mathematica, we obtain

$$a_0 = -\mu(k^2 + v^2), \quad a_2 = -(k^2 + v^2) \quad \text{and} \quad a_1 = -\beta(k^2 + v^2). \tag{11}$$

The solution of Eq. (9) reads

$$U = a_0 + a_1(G'/G) + a_2(G'/G)^2. \tag{12}$$

For $\beta^2 - 4\mu > 0$, the solution of Eq. (9) reads

$$U = a_0 + a_1\left(\frac{\frac{A}{2}\sqrt{\beta^2-4\mu}\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)+\frac{B}{2}\sqrt{\beta^2-4\mu}\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)}{A\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)+B\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)}\right) + a_2\left(\frac{\frac{A}{2}\sqrt{\beta^2-4\mu}\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)+\frac{B}{2}\sqrt{\beta^2-4\mu}\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)}{A\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)+B\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)}\right)^2. \quad (13)$$

The solution of Eq. (6) given the flux as

$$\phi = \frac{1}{2}\ln\left[a_0 + a_1\sqrt{\beta^2-4\mu}\left(\frac{\frac{A}{2}\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)+\frac{B}{2}\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)}{A\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)+B\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)}\right) + a_2\sqrt{\beta^2-4\mu}\left(\frac{\frac{A}{2}\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)+\frac{B}{2}\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)}{A\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)+B\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)}\right)^2\right]. \quad (14)$$

For the special values $k = 1$, $\beta = B = 0$ and $\mu < 0$, we find the solution obtained in [1] for Liouville equation. Moreover, Eq. (14) corresponds to an exact solution for a force-free magnetic field with

$$B_z = \left[a_2\sqrt{\beta^2-4\mu}\left(\frac{\frac{A}{2}\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)+\frac{B}{2}\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)}{A\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)+B\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)}\right)^2\right]^{\frac{1}{2}} + a_0 + a_1\sqrt{\beta^2-4\mu}\left(\frac{\frac{A}{2}\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)+\frac{B}{2}\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)}{A\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)+B\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)}\right) \quad (15)$$

However for $\beta^2 - 4\mu < 0$, the solution of Eq. (9) reads

$$U = a_0 + a_1\left(\frac{\frac{-A}{2}\sqrt{\beta^2-4\mu}\sin\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)+\frac{B}{2}\sqrt{\beta^2-4\mu}\cos\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)}{A\cos\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)+B\sin\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)}\right) + a_2\left(\frac{\frac{-A}{2}\sqrt{\beta^2-4\mu}\sin\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)+\frac{B}{2}\sqrt{\beta^2-4\mu}\cos\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)}{A\cos\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)+B\sin\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)}\right)^2. \tag{16}$$

The solution of Eq. (6) gives the flux as

$$\phi = \frac{1}{2}\ln\left[a_2\sqrt{\beta^2-4\mu}\left(\frac{\frac{-A}{2}\sin\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)+\frac{B}{2}\cos\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)}{A\cos\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)+B\sin\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)}\right)^2 + a_0 + a_1\sqrt{4\mu-\beta^2}\left(\frac{\frac{-A}{2}\sin\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)+\frac{B}{2}\cos\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)}{A\cos\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)+B\sin\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)}\right)\right]. \tag{17}$$

Equation (17) corresponds to an exact solution for a force-free magnetic field with

$$B_z = \left[a_0 + a_1\sqrt{4\mu-\beta^2}\left(\frac{\frac{A}{2}\sinh\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)+\frac{B}{2}\cosh\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)}{A\cosh\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)+B\sinh\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)}\right) + a_2\sqrt{4\mu-\beta^2}\left(\frac{\frac{A}{2}\sinh\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)+\frac{B}{2}\cosh\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)}{A\cosh\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)+B\sinh\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)}\right)^2\right]^{\frac{1}{2}}. \tag{18}$$

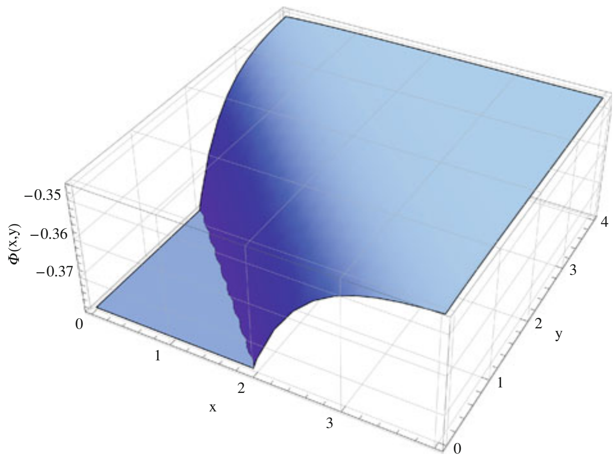

Fig. 1 The flux function $\phi(x, y)$ at $k = \nu = 1 = -\beta$ and $B = 0$ for Eq. (14)

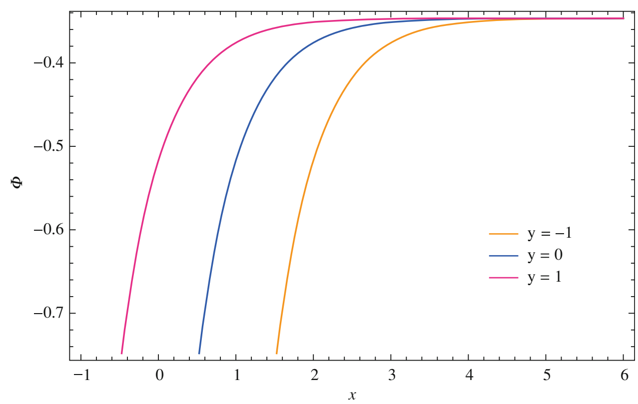

Fig. 2 The flux function ($y = -1, 0, 1$) at $k = \nu = 1 = -\beta$ and $B = 0$ for Eq. (14)

For the case $\beta^2 - 4\mu = 0$, the solution of Eq. (9) reads

$$U = a_0 + a_1\left(\frac{B}{A + B\zeta}\right) + a_2\left(\frac{B}{A + B\zeta}\right)^2. \tag{19}$$

The solution of Eq. (6) given the flux as

$$\phi = \frac{1}{2}\ln\left[a_0 + a_1\left(\frac{B}{A + B(kx + \nu y)}\right) + a_2\left(\frac{B}{A + B(kx + \nu y)}\right)^2\right]. \tag{20}$$

Equation (20) corresponds to an exact solution for a force-free magnetic field with

$$B_z = \left[a_0 + a_1\left(\frac{B}{A + B(kx + \nu y)}\right) + a_2\left(\frac{B}{A + B(kx + \nu y)}\right)^2\right]^{\frac{1}{2}}. \tag{21}$$

Moreover, for a_0, a_1 and a_2, $\alpha = \partial_\phi B_z = B_z$. The B_x and B_y components becomes

$$B_x = \partial_y \phi = \frac{-k}{\nu} B_y = \frac{k}{\nu}\partial_x \phi.$$

Figures 1–3 show the magnetic flux function, samples of streamlines of the magnetic flux function (y = constant), samples of streamlines of the magnetic flux function (x = constant) and Contour plot of the magnetic flux function corresponding to the solution of Eq. (14), respectively. Moreover, Figs. 5–7 and Figs. 8–10, with values of parameters listed in their captions, show the force-free magnetic field, samples of streamlines of the force-free magnetic field (y = constant) and samples of streamlines of the force-free magnetic field (x = constant) corresponding to the solution of Eqs. (15) and (18), respectively.

4. Two-dimensional force-free magnetic fields described by Sinh-Poisson equation

If we take the choice $B_z = \lambda\sqrt{2\cosh\phi}$, Eq. (1) turns into the nonlinear sinh-Poisson equation

$$\nabla^2\phi = -\lambda^2\sinh(\phi), \tag{22}$$

taking the transformation

$$e^\phi = u \quad \text{where} \quad \sinh(\phi) = \frac{e^\phi - e^{-\phi}}{2}.$$

Equation (22) tends to

$$2(u_x)^2 + 2(u_y)^2 - 2uu_{xx} - 2uu_{yy} + \lambda^2(u^3 - u) = 0. \tag{23}$$

Using the wave variable

$$\zeta = kx + \nu y, \quad u(x, y) = U(\zeta)$$

turns the PDE given by Eq. (23) into the ODE

$$2(k^2 + \nu^2)UU'' - 2(k^2 + \nu^2)(U')^2 - \lambda^2(U^3 - U) = 0. \tag{24}$$

Proceeding as in the previous case we obtain

$$a_2 = -4\frac{(k^2 + \nu^2)}{\lambda^2},$$

$$\beta = -\frac{\sqrt{(k^2 + \nu^2)(4\mu(k^2 + \nu^2) - \lambda^2)}}{(k^2 + \nu^2)},$$

$$a_1 = \frac{4\sqrt{(k^2 + \nu^2)(4\mu(k^2 + \nu^2) - \lambda^2)}}{\lambda^2} \quad \text{and}$$

$$a_0 = \frac{\lambda^2 - 4\mu(k^2 + \nu^2)}{\lambda^2}, \tag{25}$$

or

$$a_2 = -4\frac{k^2+v^2}{\lambda^2},$$
$$\beta = -\frac{\sqrt{(k^2+v^2)(\lambda^2+4\mu(k^2+v^2))}}{(k^2+v^2)},$$
$$a_1 = \frac{4\sqrt{(k^2+v^2)(\lambda^2+4\mu(k^2+v^2))}}{\lambda^2} \quad \text{and}$$
$$a_0 = -\frac{\lambda^2+4\mu(k^2+v^2)}{\lambda^2}, \tag{26}$$

using Eqs. (25) and (26), the solution of Eq. (24) reads

$$U = a_0 + a_1(G'/G) + a_2(G'/G)^2. \tag{27}$$

Three cases are arises

case (a) for $\beta^2 - 4\mu > 0$, the solution of Eq. (24) reads

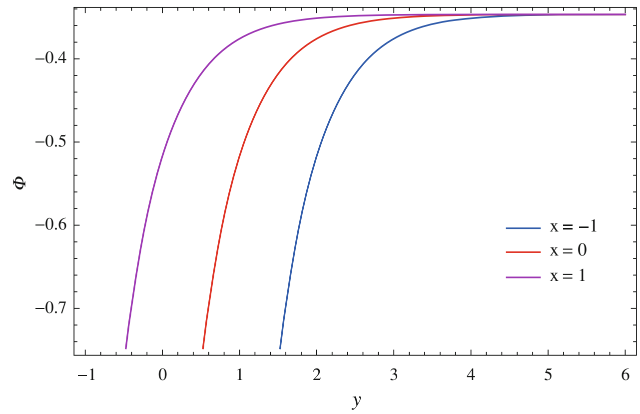

Fig. 3 The flux function ($x = -1, 0, 1$) at $k = v = 1 = -\beta$ and $B = 0$ for Eq. (14)

$$U = a_0 + a_1\left(\frac{\frac{A}{2}\sqrt{\beta^2-4\mu}\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right) + \frac{B}{2}\sqrt{\beta^2-4\mu}\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)}{A\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right) + B\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)}\right) + a_2\left(\frac{\frac{A}{2}\sqrt{\beta^2-4\mu}\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right) + \frac{B}{2}\sqrt{\beta^2-4\mu}\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)}{A\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right) + B\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)}\right)^2. \tag{28}$$

The solution of Eq. (22) gives the flux as

$$\phi = \ln\left[a_0 + a_1\sqrt{\beta^2-4\mu}\left(\frac{\frac{A}{2}\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right) + \frac{B}{2}\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)}{A\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right) + B\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)}\right) + a_2\sqrt{\beta^2-4\mu}\left(\frac{\frac{A}{2}\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right) + \frac{B}{2}\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)}{A\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right) + B\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)}\right)^2\right]. \tag{29}$$

Equation (29) corresponds to an exact solution for a force-free magnetic field with

$$B_z = \lambda \times \left[2\cosh \ln \left\langle a_2\sqrt{\beta^2-4\mu}\left(\frac{\frac{A}{2}\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)+\frac{B}{2}\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)}{A\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)+B\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)}\right)^2 + a_0 + a_1\sqrt{\beta^2-4\mu}\left(\frac{\frac{A}{2}\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)+\frac{B}{2}\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)}{A\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)+B\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}(kx+vy)\right)}\right)\right\rangle\right]^{\frac{1}{2}}. \tag{30}$$

case (b) for $\beta^2 - 4\mu < 0$, the solution of Eq. (22) given the flux as

$$\phi = \ln\left[a_2\sqrt{4\mu-\beta^2}\left(\frac{\frac{-A}{2}\sin\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)+\frac{B}{2}\cos\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)}{A\cos\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)+B\sin\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)}\right)^2 + a_0 + a_1\sqrt{4\mu-\beta^2}\left(\frac{\frac{-A}{2}\sin\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)+\frac{B}{2}\cos\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)}{A\cos\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)+B\sin\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)}\right)\right] \tag{31}$$

For the special values $a_2 = -1$, $\beta^2 - 4\mu = -4$ and $a_0 = a_1 = 0$, we find the solution obtained in Khater et al. [1] for Sinh-Poisson equation. Also, Eq. (31) corresponds to an exact solution for a force-free magnetic field with

$$B_z = \lambda \times \left[2\cosh \ln \left[a_2\sqrt{4\mu-\beta^2} \times \left(\frac{\frac{-A}{2}\sin\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)+\frac{B}{2}\cos\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)}{A\cos\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)+B\sin\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)}\right)^2 + a_0 + a_1\sqrt{4\mu-\beta^2}\left(\frac{\frac{-A}{2}\sin\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)+\frac{B}{2}\cos\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)}{A\cos\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)+B\sin\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)}\right)\right]\right]^{\frac{1}{2}}. \tag{32}$$

case (c) for $\beta^2 - 4\mu = 0$, the solution of Eq. (22) reads

$$U = a_0 + a_1\left(\frac{B}{A+B\zeta}\right) + a_2\left(\frac{B}{A+B\zeta}\right)^2. \quad (33)$$

The solution of Eq. (22) gives the flux as

$$\phi = \ln\left[a_0 + a_1\left(\frac{B}{A+B(kx+vy)}\right) + a_2\left(\frac{B}{A+B(kx+vy)}\right)^2\right]. \quad (34)$$

Equation (34) corresponds to an exact solution for a force-free magnetic field with

$$B_z = \lambda\sqrt{2\cosh\ln\left[a_0 + a_1\left(\frac{B}{A+B(kx+vy)}\right) + a_2\left(\frac{B}{A+B(kx+vy)}\right)^2\right]}. \quad (35)$$

Moreover $\alpha = (\lambda\sinh\phi)/\sqrt{2\cosh\phi}$. Thus α becomes zero for $\phi = 0$, and it becomes infinite when ϕ does. B_z becomes singular when ϕ does. The B_x and B_y components becomes

$$B_x = \partial_y\phi = \frac{-k}{v}B_y = \frac{k}{v}\partial_x\phi$$

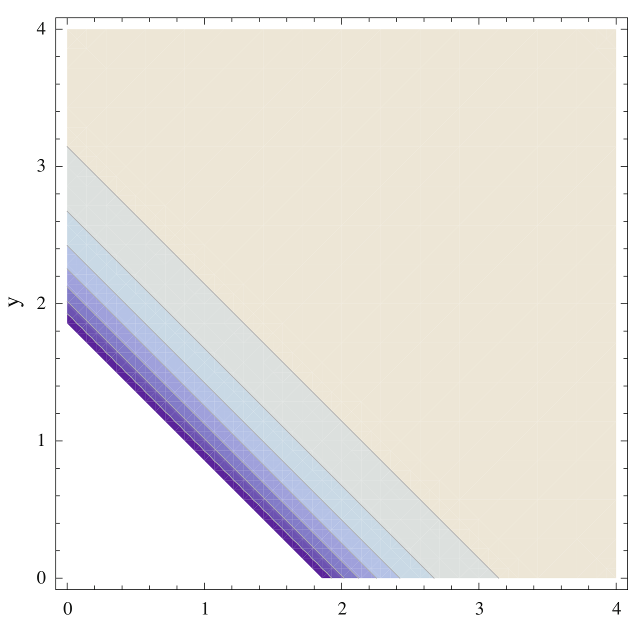

Fig. 4 The contour plot of $\phi(x, y)$ at $k = v = 1 = -\beta$ and $B = 0$ for Eq. (14)

5. Two-dimensional force-free magnetic fields described by Sine-Poisson equation

Taking the choice $B_z = \lambda\sqrt{2\cos\phi}$, Eq. (1) turns into the nonlinear sin-Poisson Equation

$$\nabla^2\phi = \lambda^2\sin(\phi). \quad (36)$$

Taking the transformation

$$e^{i\phi} = u \quad \text{where} \quad \sin(\phi) = \frac{e^{i\phi} - e^{-i\phi}}{2i}.$$

Equation (36) tends to

$$2(u_x)^2 + 2(u_y)^2 - 2uu_{xx} - 2uu_{yy} + \lambda^2(u^3 - u) = 0. \quad (37)$$

Using the wave variable

$$\zeta = kx + vy, \quad u(x,y) = U(\zeta),$$

turns the PDE given by Eq. (37) into the ODE

$$2(k^2+v^2)UU'' - 2(k^2+v^2)(U')^2 - \lambda^2(U^3 - U) = 0. \quad (38)$$

Proceeding as in the previous case we obtain

$$a_2 = -4\frac{(k^2+v^2)}{\lambda^2},$$
$$\beta = -\frac{\sqrt{(k^2+v^2)(4\mu(k^2+v^2) - \lambda^2)}}{(k^2+v^2)},$$
$$a_1 = \frac{4\sqrt{(k^2+v^2)(4\mu(k^2+v^2) - \lambda^2)}}{\lambda^2} \quad \text{and} \quad (39)$$
$$a_0 = \frac{\lambda^2 - 4\mu(k^2+v^2)}{\lambda^2},$$

or

$$a_2 = -4\frac{k^2+v^2}{\lambda^2},$$
$$\beta = -\frac{\sqrt{(k^2+v^2)(\lambda^2 + 4\mu(k^2+v^2))}}{(k^2+v^2)},$$
$$a_1 = \frac{4\sqrt{(k^2+v^2)(\lambda^2 + 4\mu(k^2+v^2))}}{\lambda^2} \quad \text{and} \quad (40)$$
$$a_0 = -\frac{\lambda^2 + 4\mu(k^2+v^2)}{\lambda^2},$$

using Eqs. (39) and (40) the solution of Eq. (38) reads

$$U = a_0 + a_1(G'/G) + a_2(G'/G)^2. \quad (41)$$

Three cases may be considered:

case (a) for $\beta^2 - 4\mu > 0$, the solution of Eq. (36) reads

$$u = a_0 + a_1\left(\frac{\frac{A}{2}\sqrt{\beta^2-4\mu}\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right) + \frac{B}{2}\sqrt{\beta^2-4\mu}\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)}{A\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right) + B\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)}\right) + a_2\left(\frac{\frac{A}{2}\sqrt{\beta^2-4\mu}\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right) + \frac{B}{2}\sqrt{\beta^2-4\mu}\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)}{A\cosh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right) + B\sinh\left(\frac{\sqrt{\beta^2-4\mu}}{2}\zeta\right)}\right)^2. \quad (42)$$

case (b) for $\beta^2 - 4\mu < 0$

$$u = \left[a_2\sqrt{4\mu-\beta^2}\left(\frac{\frac{-A}{2}\sin\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right) + \frac{B}{2}\cos\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)}{A\cos\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right) + B\sin\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)}\right)^2 + a_0 + a_1\sqrt{4\mu-\beta^2}\left(\frac{\frac{-A}{2}\sin\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right) + \frac{B}{2}\cos\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)}{A\cos\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right) + B\sin\left(\frac{\sqrt{4\mu-\beta^2}}{2}(kx+vy)\right)}\right)\right]. \quad (43)$$

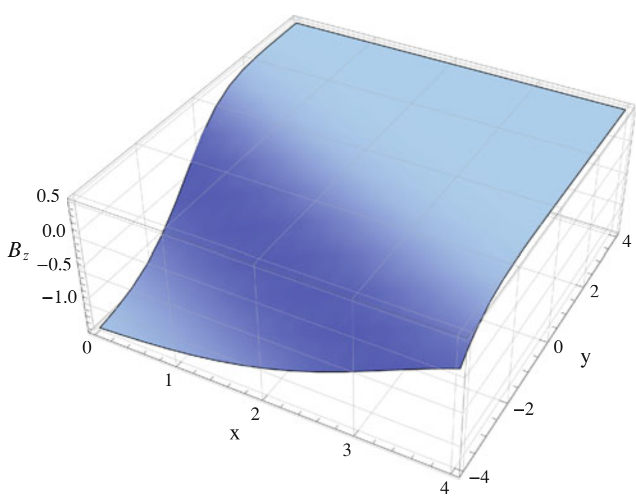

Fig. 5 The force-free magnetic field $B_z(x, y)$ at $k = \nu = 1 = -\beta$ and $B = 0$ for Eq. (15)

case (c) for $\beta^2 - 4\mu = 0$

$$u = a_0 + a_1\left(\frac{B}{A+B\zeta}\right) + a_2\left(\frac{B}{A+B\zeta}\right)^2. \quad (44)$$

For the above cases, the flux is given as

$$\phi = \cos^{-1}\left(\frac{u^2+1}{2u}\right), \quad (45)$$

with a force-free magnetic field

$$B_z = \lambda\sqrt{2\cos\phi} = \lambda\sqrt{\left(\frac{u^2+1}{u}\right)}. \quad (46)$$

This component becomes singular when u is singular. Moreover $\alpha = -(\lambda\sin\phi)/\sqrt{2\cos\phi}$. Thus α becomes zero for $\phi = 0$, and it becomes infinite when ϕ =nπ, n is integer. B_z becomes singular when $u = 0$. The B_x and B_y components becomes

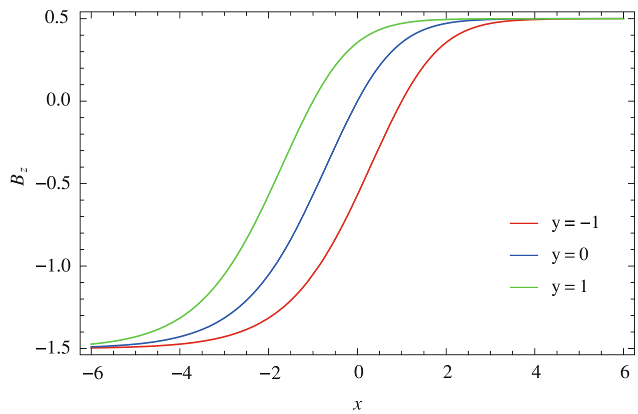

Fig. 6 The force-free magnetic field ($y = -1, 0, 1$) at $k = \nu = 1 = -\beta$ and $B = 0$ for Eq. (15)

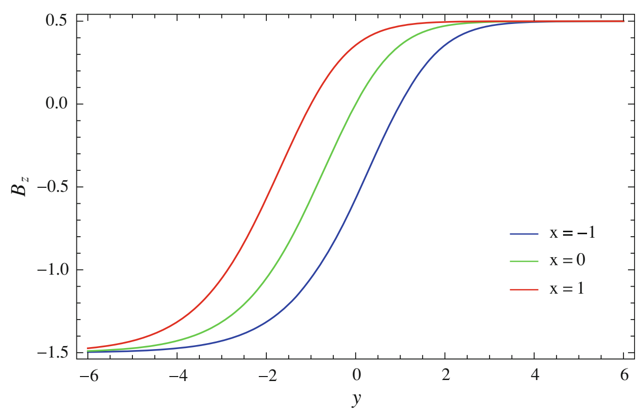

Fig. 7 The force-free magnetic field ($x = -1, 0, 1$) at $k = \nu = 1 = -\beta$ and $B = 0$ for Eq. (15)

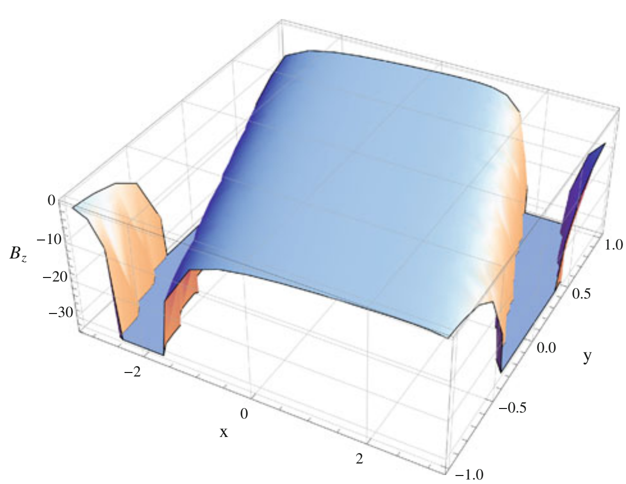

Fig. 8 The force-free magnetic field $B_z(x, y)$ at $k = \nu = \mu = \beta = 1$ and $B = 0$ for Eq. (18)

$$B_x = \partial_y \phi = \frac{-k}{\nu} B_y = \frac{k}{\nu} \partial_x \phi.$$

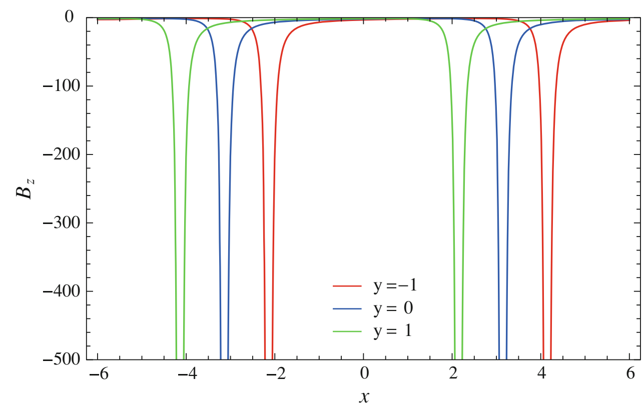

Fig. 9 The force-free magnetic field ($y = -1, 0, 1$) at $k = \nu = \mu = \beta = 1$ and $B = 0$ for Eq. (18)

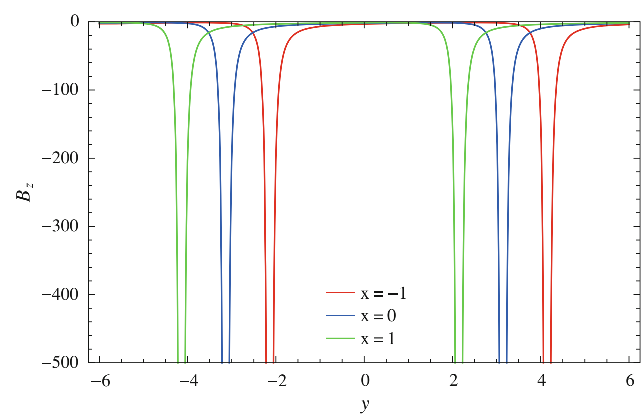

Fig. 10 The force-free magnetic field ($x = -1, 0, 1$) at $k = \nu = \mu = \beta = 1$ and $B = 0$ for Eq. (18)

As in the previous case, by taking special value of constants, we find the solution obtained for Sine-Poisson equation earlier [1], and therefore this result shows the power of our method. Also in this paper we have found several solutions which are different from the solutions in [1]. This showed also that why various plots (Figs. 1–10) of the present work are different from the plots reported earlier [1].

6. Conclusions

Using various non-linear equations we have obtained several force-free magnetic fields with non-constant α, the ratio between current density and magnetic induction. These fields may be of use in various situations, in particular in the solar or a stellar atmosphere where the coronal fields often possess an arcade-like structure, so that a two-dimensional analysis is appropriate. In most studies one has used a constant α for simplicity. However, this is usually at best an approximate physical assumption, urging for a variable α.

References

[1] A H Khater, D K Callebaut and M A Abdelkawy *Phys. Plasmas* **17** 122902 (2010)
[2] H You-Qiu *Chin. Astron. Astrophys*. **19** 229 (1995)
[3] H You-Qiu *Chin. Astron. Astrophys*. **24** 217 (2000)
[4] H You-Qiu and L Lin *Chin. Astron. Astrophys*. **26** 451 (2002)
[5] Amari et al., *Sol. Phys.* **174** 129 (1997)
[6] T Z Boulmezaoud and T Amari *Math. Comput. Model* **34** 903 (2001)
[7] E Romashets and M Vandas *Adv. Space Res.***36** 2268 (2005)
[8] A Biswas and E V Krishnan *Indian J. Phys*. **85** 1513 (2011)
[9] A H Bhrawy, M M Tharwat, A Yildirim and M A Abdelkawy et al., *Indian J. Phys*. **86** 1107 (2012)
[10] M L Wang *Phys. Lett. A*. **199** 169 (1990)
[11] M L Wang et al., *Phys. Lett. A*. **216** 67 (1996)
[12] L Yang et al., *Phys. Lett. A*.**260** 55 (1999)
[13] L Yang et al., *Phys. Lett. A*.**278** 267 (2001)
[14] E J Parkes and B R Duffy *Phys. Lett. A* **229** 217 (1997)
[15] E Fan *Phys. Lett. A*.**277** 212 (2000)
[16] H Hirota *J. Math. Phys*.**14** 810 (1973)
[17] N A Kudryashov *Phys. Lett. A* **147** 287 (1997)
[18] A Otwinowski et al., *Phys. Lett. A* **128** 483 (1988)
[19] S K Liu et al., *Appl. Math. Mech*. **22** 326 (2000)
[20] C Yan *Phys. Lett. A*. **224** 77 (1996)
[21] A V Porubov *Phys. Lett. A*. **221** 391 (1996)
[22] A V Porubov and M G Velarde *J. Math. Phys*. **40** 884 (1999)
[23] A V Porubov and D F Parker *Wave Motion*. **29** 97 (1999)
[24] M L Wang X Li and J Zhang *Phys. Lett. A* **372** 4 (2008)
[25] A Malik, F Chand, H Kumar and S C Mishra *Indian J. Phys*. **86** 129 (2012)
[26] A K Malik et al., *Appl. Math. Comput*..**216** 2596 (2010)
[27] A K Malik et al., *Pramana-J. Phys*. **78** 513 (2012)
[28] A K Malik et al., *Comput. Math. Appl*. **64** 2850 (2012)
[29] N A Kudryashov and A Pickering *J. Phys. A: Math. Gen.* **31** 9505 (1998)
[30] N A Kudryashov *J. Phys. A: Math. Gen*. **32** 999 (1999)